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THE PROBLEM OF CONSTRUCTING LYAPUNOV'S REDUCING TRANSFORMATION†

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A Lyapunov transformation for systems of linear ordinary differential equations which depend periodically on time is constructed. The transformation reduces any such system to a system with constant coefficients. The construction is based on a procedure of successive replacement of variables. Convergence conditions are established, and if convergence occurs, its rate is estimated. The estimates are used to determine whether the zero equilibrium is Lyapunov stable.

Lyapunov proposed the problem of whether a system of linear ordinary differential equations with continuous T-periodic coefficients is reducible to a system with constant coefficients. He proved that any such system is reducible [1]. With regard to a system $\dot{x}=\varepsilon A(t)x$, a reducing transformation was constructed as a formal series in powers of ε [2]; it was proved that this series converges if $\varepsilon T\max_i \sup_{t\in[0,T]}|\lambda_i(t)|<\ln 2$, where $\lambda_i(t)$ are the eigenvalues of the matrix $A(t)$ [2]. Since then [3–5], different aspects of the formal-series construction of reducing replacements have been considered.

A procedure of successive substitutions of variables has been used to find Lyapunov reducing transformations [6]. With regard to a system of the form $\dot{x}=(A+B(t))x$, where A is a constant matrix and $B(t)$ is a quasi-periodic matrix, it has been proved that convergence occurs if all the eigenvalues of A are real and strongly distinct, and the norm of $B(t)$ is sufficiently small and satisfies an infinite chain of inequalities. For purely periodic systems, however, simpler and more intuitive estimates will ensure that the procedure of successive replacements is convergent.

1. Consider the system

$$\dot{x}=A_0(t)x \tag{1.1}$$

$$(x=(x^1,\ldots,x^n),\quad A_0=\bar{A}_0+\tilde{A}_0,\quad \bar{A}_0=T^{-1}\int_0^T A_0(t)dt,\quad \tilde{A}_0=A_0-\bar{A}_0)$$

where $A_0(t)$ is a continuous T-periodic $(n\times n)$ matrix. Denote the space of continuous T-periodic $(n\times n)$ matrices by C. Define a norm in that space by

$$\|A\|_C=\max_{i=1,\ldots,n}\sum_{j=1}^n \sup_{[0,T]}|a_{ij}(t)|$$

where a_{ij} are the elements of A. With this norm the space C is complete [7]. The subscript C in the norm symbol will henceforth be omitted.

We apply a procedure of successive replacements to (1.1), where the kth replacement is defined by

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$$x_{(k-1)} = (E + Z_k)x_{(k)}, \quad k = 1, 2, \ldots, \quad x_{(0)} \equiv x \tag{1.2}$$

where $x_{(k-1)}$, $x_{(k)}$ are the old and new variables, respectively, E is the identity matrix and $Z_k(t)$ a T-periodic continuously differentiable solution of the matrix differential equation

$$\dot{Z} = \bar{A}_{k-1}Z - Z\bar{A}_{k-1} + \tilde{A}_{k-1}, \tag{1.3}$$

satisfying the condition $\bar{Z}_k = 0$. Such a solution is defined if the eigenvalues of the matrix $\bar{A}_{k-1}$ satisfy the relations

$$\lambda_i - \lambda_j \neq 2\pi T^{-1}\sqrt{-1}m, \quad i, j = 1, \ldots, n \tag{1.4}$$

for any non-zero integer m [4, 5].

A sufficient condition for the kth replacement (1.2) to be non-degenerate is [8]

$$\|Z_k\| < 1 \tag{1.5}$$

After k steps, the transformed system has the form

$$\dot{x}_{(k)} = A_k x_{(k)}, \quad A_k = \bar{A}_{k-1} + (E + Z_k)^{-1}\tilde{A}_{k-1}Z_k \tag{1.6}$$

If the first m successive replacements are defined and non-degenerate, then the matrices A_k are obviously continuous and T-periodic for $k \leqslant m$, i.e. systems (1.6) are of the same class as (1.1). We will establish conditions for the existence of non-degenerate replacements at each step and conditions for the convergence of the procedure. The intermediate results may be formulated as several lemmas.

Lemma 1. Let $A, B \in C$. Then

$$\|AB\| \leqslant \|A\| \cdot \|B\|, \quad \|\bar{A}\| \leqslant \|A\|, \quad \|\tilde{A}\| \leqslant 2\|A\| \tag{1.7}$$

If also $dB/dt \in C$, then

$$\|B - \bar{B}\| \leqslant \tfrac{1}{2}T\|dB/dt\| \tag{1.8}$$

Inequalities (1.7) follow from the properties of matrix norms [8]. An estimate analogous to (1.8) was proved in [9].

To abbreviate the subsequent operations, we introduce the notation

$$\begin{gathered} \nu_k = \|\bar{A}_k\|, \quad \mu_k = \|\tilde{A}_k\| \\ \sigma_k = \frac{T\mu_k/2}{1 - T\nu_k}, \quad \delta_k = \frac{2\sigma_k}{1 - \sigma_k - 2\sigma_k^2}, \quad k = 0, 1, \ldots \end{gathered} \tag{1.9}$$

Lemma 2. If

$$T\nu_0 < 1, \quad \sigma_0 < (\sqrt{17} - 3)/4 \tag{1.10}$$

then the replacement of variables (1.2) is defined and non-degenerate at the first step, and

$$T\nu_1 < 1, \quad \mu_1 < \mu_0\delta_0, \quad \sigma_1 < \sigma_0\delta_0, \quad \delta_1 < \delta_0^2, \quad 0 < \delta_0 < 1 \tag{1.11}$$

Proof. By the properties of matrix norms [8], if λ_j are the eigenvalues of the matrix A_0, then

$\max_{j=1,\ldots,n}|\lambda_j|<\|\bar{A}_0\|$, and by the first of conditions (1.10), $|\lambda_j|<1/T$, $(j=1,\ldots,n)$. Hence we conclude that inequalities (1.4) hold, the required solution Z_1 of Eq. (1.3) may be found, the substitution (1.2) is defined. Further, using inequality (1.8), we find that the solution Z_1 of Eq. (1.3) satisfies the estimate $\|Z_1\|\leqslant T\|\dot{Z}_1\|/2\leqslant T(2\nu_0\|Z_1\|+\mu_0)/2$, or

$$\|Z_1\|\leqslant\sigma_0 \tag{1.12}$$

Hence, using the second of conditions (1.10), we conclude that $\|Z_1\|<1$, and so, by (1.5), the replacement is indeed non-degenerate at the first step.

The second inequality in (1.10) is a necessary and sufficient condition for the truth of $0<\delta_0<1$.

Now, using the explicit form of the matrix A_1 (1.6) and estimates (1.7), we obtain

$$\nu_1\leqslant\nu_0+\mu_0\sigma_0/(1-\sigma_0),\ \ \mu_1\leqslant 2\mu_0\sigma_0/(1-\sigma_0),\ \ \sigma_1\leqslant\sigma_0\delta_0 \tag{1.13}$$

from which (1.11) follows by a direct argument, using definitions (1.9) and the condition $1-\sigma_0-2\sigma_0^2>0$.

Lemma 3. Under the assumptions of Lemma 2, the successive replacements of variables (1.2) are defined and non-degenerate at each step, i.e. one can construct them indefinitely and for all $k=1,2,\ldots$

$$T\|\bar{A}_k\|<1,\ \ \|\bar{A}_k\|=\mu_k\leqslant\mu_0\delta_0^{2^k-1},\ \ \|Z_{k+1}\|\leqslant\sigma_k\leqslant\sigma_0\delta_0^{2^k-1},\ \ \delta_k\leqslant\delta_0^{2^k} \tag{1.14}$$

The proof is by induction, using Lemma 2 and inequality (1.12).

Lemma 4. Assume that the successive T-periodic replacements (1.2) (the actual form of the matrix Z_k here is immaterial) are defined, non-degenerate and continuously differentiable for each $k=1,2,\ldots$, and that there constants M_1, $M_2>0$ exist such that

$$\sum_{k=1}^{\infty}\|Z_k\|\leqslant M_1,\ \ \sum_{k=1}^{\infty}\|\dot{Z}_k\|\leqslant M_2 \tag{1.15}$$

Then the procedure of successive replacements of variables is convergent, i.e. a matrix $Q_\infty\in C$ exists such that

$$Q_\infty=\lim_{k\to+\infty}Q_k,\ \ Q_k=(E+Z_1)\ldots(E+Z_k) \tag{1.16}$$

In addition, the substitution $x=Q_\infty y$ transforms system (1.1) to a system $\dot{y}=A_\infty y$, where

$$A_\infty=\lim_{k\to+\infty}A_k,\ \ A_k=Q_k^{-1}(A_0Q_k-\dot{Q}_k) \tag{1.17}$$

A_k being the matrix defining the transformed system after k iterations.

Proof. The chain of inequalities

$$\|E+Z_k\|\leqslant 1+\|Z_k\|,\ \ \ln\|Q_k\|\leqslant\ln\prod_{j=1}^{k}\left(1+\|Z_j\|\right)\leqslant\sum_{j=1}^{k}\|Z_j\|\leqslant M_1$$

implies that for arbitrary m and k

$$\|Q_{m+k}-Q_k\|\leqslant\|Q_k\|\ln\prod_{j=1}^{m}\left(1+\|Z_{k+j}\|\right)\leqslant\sum_{j=1}^{m}\|Z_{k+j}\|\exp M_1$$

Hence, by the convergence of the first series in (1.15), we deduce that the matrices Q_k form a fundamental sequence which, as C is a complete space, converges to some matrix $Q_\infty\in C$. The proof that the sequence $\dot{Q}_k$ is convergent is similar. Further, since the elements of the matrices Q_k, $\dot{Q}_k$ are uniformly convergent, we can differentiate term by term, concluding that Q_∞ is a continuously differentiable matrix and $\dot{Q}_\infty=\lim_{k\to+\infty}\dot{Q}_k$. Thus, the limit transformation is defined. That it is non-degenerate follows from the fact that the replacements of variables at all steps are non-degenerate and that the first series (1.15) is convergent, hence $\|Z_k\| \to 0$, $\det(E+Z_k) \to 1$ as $k \to +\infty$.

Thus, the matrix Q_∞ defines a certain non-degenerate replacement of variables and, using the formula for A_k in (1.17), we conclude that $A_k \to A_\infty = Q_\infty^{-1}(A_0 Q_\infty - \dot{Q}_\infty)$, $k \to +\infty$.

We now observe that, under the assumptions of Lemma 3, the assumptions of Lemma 4 will also hold for the successive replacements defined by (1.2) and (1.3), since by (1.14)

$$\sum_{k=1}^{\infty} \|Z_k\| \leq \sum_{k=1}^{\infty} \sigma_k \leq \sigma_0 \sum_{k=1}^{\infty} \delta_0^{2^k-1} < \infty$$

$$\sum_{k=1}^{\infty} \|\dot{Z}_k\| \leq \sum_{k=1}^{\infty} \left(2\|Z_k\|\nu_k + \mu_k\right) \leq 2\sum_{k=1}^{\infty} \sigma_k / T < \infty$$

(the derivation of the second chain of inequalities uses Eq. (1.3) for the matrix Z_k and definitions (1.9)). Consequently, under the assumptions of Lemma 3 the procedure is indeed convergent. It now follows from the estimates (1.14) that $\|A_\infty - \overline{A}_\infty\| = \lim_{k\to+\infty} \|A_k - \overline{A}_k\| = 0$, i.e. the limit matrix A_∞ is constant. Thus, the limit transformation is a Lyapunov reducing transformation.

The final result may be stated as a theorem.

Theorem 1. Suppose that the T-periodic matrix A_0 defining system (1.1) satisfies the conditions

$$T\|\overline{A}_0\| < 1, \quad T\|\tilde{A}_0\| / \left(1 - T\|\overline{A}_0\|\right) < \left(\sqrt{17} - 3\right)/2 \tag{1.18}$$

Then

1. The successive replacements of variables (1.2) and (1.3) are defined, non-degenerate and T-periodic at each step $k = 1, 2, \ldots$.

2. The procedure of successive replacements is convergent and the limiting transformation $x = Q_\infty y$ (where Q_∞ is the matrix defined by (1.16)) is a Lyapunov reducing transformation, i.e. $A_\infty = Q_\infty^{-1}(A_0 Q_\infty - \dot{Q}_\infty)$ is a constant matrix.

The above procedure enables one constructively to approximate a Lyapunov reducing transformation. Using the estimates (1.14) of Lemma 3, there is no difficulty in estimating the rate of convergence of the successive transformations and of the sequences of matrices A_k, $\overline{A}_k$

$$\begin{gathered}
\|Q_\infty - Q_k\| \leq (1+\sigma_0)^{D(\delta_0)} \sigma_0 \delta_0^{2^k-1} D\left(\delta_0^{2^k}\right) \\
\|A_\infty - A_k\| \leq \mu_0 \delta_0^{2^k-1}\left[D\left(\delta_0^{2^k}\right) + \tfrac{1}{2}\delta_0^{2^k} D\left(\delta_0^{2^{k+1}}\right)\right] \\
\|A_\infty - \overline{A}_k\| \leq \tfrac{1}{2}\mu_0 \delta_0^{2^{k+1}-1} D\left(\delta_0^{2^{k+1}}\right), \quad k = 0, 1, 2, \ldots \\
\left(D(x) = \sum_{m=0}^{\infty} x^{2^m - 1}\right)
\end{gathered} \tag{1.19}$$

2. The conditions stated in Theorem 1 for the convergence of the successive replacements may frequently be improved. In particular, this may be done by narrowing down the permissible class of matrices, slightly modifying the norm.

Let M denote the space of T-periodic matrices whose elements may be expressed as absolutely convergent Fourier series. Define a norm in M by

$$\|A\|_M = \sum_{m=-\infty}^{+\infty} \max_{i=1,\ldots,n} \sum_{j=1}^{n} \left|a_{ij}^m(t)\right|$$

where a_{ij}^m are the Fourier coefficients of the expansion in series of the element a_{ij} of the matrix

A. Clearly, $M \subset C$ and $\|A\|_M \geqslant \|A\|_C$ for any matrix $A \in M$, that is, the norm $\|\cdot\|_M$ is as a rule "worse" than $\|\cdot\|_C$. However, this makes it possible to improve the conditions for the substitutions (1.18) to converge, which then have the simple form

$$T\|A_0\|_M < \pi \tag{2.1}$$

In all other respects the formulation of Theorem 1 remains the same.

The proof proceeds in exactly the same way as for the space C, relying on a few modifications in the estimates of Lemmas 1 and 2 (we omit the subscript M in the norm symbol)

$$\begin{aligned}
&(1.7)\colon \|A\| = \|\bar{A}\| + \|\tilde{A}\| \\
&(1.8)\colon \|B - \bar{B}\| \leqslant \frac{T}{2\pi}\left\|\frac{dB}{dt}\right\| \\
&(1.9)\colon \sigma_k = \frac{T\mu_k/2}{\pi - T\nu_k}, \quad \delta_k = \frac{\sigma_k}{1-\sigma_k} \\
&(1.10)\colon T\nu_1 < \pi, \ \sigma_0 < 1/2 \\
&(1.13)\colon \nu_1 + \mu_1 \leqslant \nu_0 + \mu_0\sigma_0/(1-\sigma_0) \\
&\mu_1 \leqslant \mu_0\sigma_0/(1-\sigma_0), \quad \sigma_1 \leqslant \sigma_0\delta_0
\end{aligned} \tag{2.2}$$

(the number of the old formula is specified on the left; the modified formula appears on the right).

Note that the convergence condition (2.1) improves the sufficient convergence condition of [5] which, for systems $\dot{x} = (A_0 + \varepsilon B_0(t))x$, where $A_0 = \text{const}$, is the inequality $\varepsilon T\|B_0\|(\pi - T\|A_0\|) < 2(3 - 2\sqrt{(2)})$.

Some changes are also necessary in the estimates for the convergence rate of the sequences of transformed matrices—the coefficient 1/2 in the second and third inequality of (1.19) is to be replaced by unity.

3. On the basis of Theorem 1 and the convergence rate estimates, one can establish sufficient conditions for the equilibrium position $x = 0$ of system (1.1) to be asymptotically Lyapunov stable (unstable). Indeed, in the convergent case, the last estimate of (1.19) or its analogue in Section 2 define a guaranteed neighbourhood of the matrix $\bar{A}_k$ that will contain the matrix A_∞. If $\bar{A}_k$ is such that all the constant matrices in that neighbourhood have eigenvalues with only negative real parts (an eigenvalue exists with a positive real part), then this is true, in particular, for A_∞. Consequently, the equilibrium $y = 0$ of the reduced system $\dot{y} = A_\infty y$ will be uniformly asymptotically Lyapunov stable (unstable). Moreover, since the reduced system was derived from the first by a linear continuous T-periodic transformation, this conclusion is also true for the equilibrium $x = 0$ of system (1.1).

4. *Example. The Stability of the upper equilibrium position of a pendulum suspended from a vibrating point, taking viscous friction into account.* The equation of motion of a pendulum whose point of suspension is experiencing vertical sinusoidal oscillations of amplitude b and frequency ω, allowing for viscous friction, is

$$\ddot{\phi} + \nu\dot{\phi} + (g - b\omega^2 \sin\omega t)l^{-1}\sin\phi = 0 \tag{4.1}$$

Here ϕ is the deviation of the pendulum from the vertical, ν is the coefficient of viscous friction, g is the acceleration due to gravity and l is the length of the pendulum.

We will establish the conditions under which the upper equilibrium position of the pendulum is uniformly asymptotically stable in the linear approximation; hence, by Lyapunov's theorem on asymptotic stability in the first approximation, it is also uniformly asymptotically stable for the complete system.

Put

$$\tau = t\sqrt{g/l}, \quad \Omega = \omega\sqrt{g/l}, \quad \kappa = b\omega/\sqrt{gl}, \quad f = \nu\sqrt{g/l}$$

and linearize the equation of motion (4.1) in the neighbourhood of the equilibrium position $\phi = \pi$. This gives (the prime denotes differentiation with respect to τ)

$$\phi'' + f\phi' - (1 - \kappa\Omega\sin\Omega\tau)\phi = 0$$

We now replace this second-order equation by a system of two linear equations in the variables ϕ and ϕ'' and substitute

$$\begin{Vmatrix}\phi\\ \phi'\end{Vmatrix} = \begin{Vmatrix}1 & 0\\ \chi & 0\end{Vmatrix}\begin{Vmatrix}x_1\\ x_2\end{Vmatrix} \quad (\chi = \kappa\cos\Omega\tau)$$

This gives a system of type (1.1)

$$\begin{Vmatrix}x_1\\ x_2\end{Vmatrix}' = A_0\begin{Vmatrix}x_1\\ x_2\end{Vmatrix}, \quad A_0 = \begin{Vmatrix}\chi & 1\\ \chi^2 - \chi f & -\chi - f\end{Vmatrix}$$

where A_0 is a matrix in the space M.

The condition for the procedure of successive replacements (2.1) to converge in this case takes the form

$$\max(1, |\kappa^2/2 - 1| + f) + \kappa(1 + \kappa/2 + f) < \Omega/2 \tag{4.2}$$

We will now establish the conditions for asymptotic stability of the equilibrium. Let $A_\infty = \overline{A}_0 + a$, where a is some constant matrix. Then estimate (2.3) for $k = 0$ may be written in the form

$$\|a\| < \mu_0\delta_0 D(\delta_0^2) = \varepsilon$$

where δ_0 and $D(x)$ are defined by (2.2) and the last expression in (1.19), respectively, and

$$\nu_0 = \max(1, |\kappa^2/2 - 1| + f), \quad \mu_0 = \kappa(1 + \kappa/2 + f) \tag{4.3}$$

The condition for A_∞ to have only eigenvalues with negative real parts may be derived from the Routh–Hurwitz criterion

$$-f + a_{11} + a_{22} < 0, \quad a_{11}(-f + a_{22}) - (1 + a_{12})(1 - \kappa^2/2 + a_{21}) > 0$$

where a_{ij} are the elements of a. These inequalities will certainly hold if for $\varepsilon < 1$

$$-f + 2\varepsilon < 0, \quad -1 + (1 - \varepsilon)\kappa^2/2 - f\varepsilon > 0, \quad 0 < \varepsilon < 1 \tag{4.4}$$

Thus, if inequalities (4.2) and (4.4) hold, the equilibrium $\phi = \pi$ of the pendulum is uniformly asymptotically Lyapunov stable.

It is obvious that in the case of constant κ, f and $\Omega \to \infty$ ($\varepsilon \to 0$), i.e. in the zeroth approximation with respect to $1/\Omega$, these conditions are identical with the more familiar condition $b\omega/\sqrt{(gl)} > \sqrt{(2)}$ (see [10]).

In the linear approximation with respect to $1/\Omega$, these conditions may be written in the form

$$\nu_0 + \mu_0 < \Omega/2, \quad f - 2\mu_0^2/\Omega > 0$$

$$-1 + (1 - \mu_0^2/\Omega)\kappa^2/2 - f\mu_0^2/\Omega > 0$$

where ν_0 and μ_0 are defined by (4.3).

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